

Analytic Functions and the Cauchy–Riemann Equations

Complex Differentiability and the Cauchy–Riemann Equations

Lecture Notes

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1 Analytic Functions

Definition (complex differentiability): A function f is (*complex*) *differentiable* at z_0 if the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Important — where is f defined? For this to be interesting, f should be defined on a whole disc $B(z_0, r)$ for some $r > 0$: unlike on the real line, the point z is free to approach z_0 from every direction of the plane, and the limit must be the same for all of them.

Equivalent definition: f is differentiable at z_0 if and only if

$$f(z) = f(z_0) + (z - z_0) \varphi(z), \quad \text{where } \varphi \text{ is continuous at } z_0,$$

in which case $\varphi(z_0) = f'(z_0)$.

Proof (of equivalence). ($\Leftarrow$) If the representation holds, then $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \varphi(z) = \varphi(z_0)$, so $f'(z_0)$ exists and equals $\varphi(z_0)$.

($\Rightarrow$) Conversely, take

$$\varphi(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0, \\ f'(z_0), & z = z_0; \end{cases}$$

continuity of φ at z_0 is precisely the statement that f is differentiable there. $\square$

Analytic (holomorphic) functions

f is called **analytic** (or **holomorphic**) at a point $z_0 \in \mathbb{C}$ if for some $\delta > 0$ it has a derivative at *every* point $z \in B(z_0, \delta)$. A function analytic at every point of an open set is analytic there; a function analytic on all of $\mathbb{C}$ is called **entire**.

Remark — one point is not enough. Differentiability at a single point (or even along a curve) is *not* interesting. For instance, $F(z) = \operatorname{Re} z |\operatorname{Re} z|$ is complex differentiable at every point of the line $\operatorname{Re} z = 0$ and nowhere else; in particular it is analytic *nowhere*. What matters is differentiability at every point of a whole neighborhood.

Arithmetic (the same as in Calculus). If $f'(z)$ and $g'(z)$ exist, then $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$, and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ where $g(z) \neq 0$. If $f'(z)$ exists and g' exists at $f(z)$, then $(g(f(z)))' = g'(f(z)) \cdot f'(z)$ (Chain Rule). The proofs are identical to the real case.

2 Real vs. Complex Differentiability

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ through $z = x + iy$, a function $f = u + iv$ becomes a map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Recall that f is **real-differentiable** at z_0 if there is a real-linear map T with

$$f(z_0 + h) = f(z_0) + T(h) + o(|h|), \quad h \rightarrow 0.$$

By contrast, **complex differentiability** demands the sharper estimate

$$\lim_{h \rightarrow 0} \frac{|f(z_0 + h) - f(z_0) - f'(z_0)h|}{|h|} = 0,$$

i.e. the increment is approximated not merely by a real-linear map but by *multiplication by the complex number $f'(z_0)$* .

From calculus, the differential of $f = u + iv$ is

$$T(h) = \frac{\partial f}{\partial x} \xi + \frac{\partial f}{\partial y} \eta = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (\text{real form}), \quad h = \xi + i\eta.$$

Substituting $\xi = \frac{h+\bar{h}}{2}$ and $\eta = \frac{h-\bar{h}}{2i}$ and collecting the coefficients of h and $\bar{h}$ gives the complex form

$$T(h) = \frac{\partial f}{\partial z}(z_0)h + \frac{\partial f}{\partial \bar{z}}(z_0)\bar{h},$$

where the coefficients are the **Wirtinger derivatives**

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

3 The Cauchy–Riemann Equations

Dividing the real-differentiable expansion by h and letting $h \rightarrow 0$,

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \rightarrow 0} \frac{T(h)}{h} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \lim_{h \rightarrow 0} \frac{\bar{h}}{h}.$$

The last limit does *not* exist ($\bar{h}/h = 1$ along the real axis, -1 along the imaginary axis), so the whole limit exists precisely when the offending term vanishes. This proves:

Theorem (Cauchy–Riemann)

Let f be a real-differentiable function at z_0 . Then f is complex differentiable at z_0 **if and only if**

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

In that case $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

Warning: real-differentiability is a standing assumption. The equivalence above is only as good as its hypothesis: it presupposes that f is already real-differentiable at z_0 . Dropping that hypothesis and keeping only the Cauchy–Riemann equations is *not* enough to conclude analyticity in general – this subtlety (and the Looman–Menchoff theorem) is discussed in a later note.



Wilhelm Wirtinger
(1865–1945)



Augustin-Louis Cauchy
(1789–1857)



Bernhard Riemann
(1826–1866)

Writing $f = u + iv$ and expanding $\partial f / \partial \bar{z} = 0$ via the Wirtinger formula,

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) = 0,$$

and separating real and imaginary parts yields the classical system.

The Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Matrix interpretation. The Cauchy–Riemann equations say exactly that the Jacobian of f has the special form

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} = M_{f'(z_0)},$$

the matrix of multiplication by the complex number $f'(z_0)$. In other words, complex differentiability means the differential — a priori only real-linear — is in fact complex-linear. Consequently the derivative can be read off from either component alone:

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}.$$

4 A Real Analysis Refresher

A common trap is to believe that the mere *existence* of the partial derivatives is enough. Before reading on, try the quiz.

Question

Suppose $f = u + iv : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has all first partial derivatives at every point $z \in B(z_0, r)$. What can you say about f ?

1. f is continuous in $B(z_0, r)$, but not always differentiable.
2. f is differentiable in $B(z_0, r)$.
3. The functions u and v are differentiable in $B(z_0, r)$.
4. f can be discontinuous at some points of $B(z_0, r)$.

The correct answer is **(4)**: the existence of partials does not even force continuity.

Counterexample. With $z = x + iy$, let

$$f(z) = \begin{cases} \frac{xy}{x^2 + y^2} + i \cdot 0, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

On both coordinate axes $f \equiv 0$, so all four partials vanish at the origin,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \Big|_0 = 0,$$

and the Cauchy–Riemann equations hold at $z = 0$. Yet f is **not continuous** at 0: along the diagonal $z = t(1 + i)$,

$$f(z) = \frac{t^2}{2t^2} = \frac{1}{2} \not\rightarrow 0.$$

What rescues the situation is *continuity of the partials*.